1 Integrating Term by Term

Theorem 1 *Consider the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \; |z| < R (R \neq 0)$$

Let C be a simple piecewise smooth curve which lies inside the circle of convergence. Then we can ***integrate the power series term by term:***

$$\int_C \left(\sum_{n=0}^{\infty} a_n z^n \right) dz = \sum_{n=0}^{\infty} a_n \int_C z^n dz \tag{1}$$

Proof. The function $f(z)$ defined by the power series is continuous on C, so the integrals in (1) are well-defined. We need to show that

$$\lim_{n \to \infty} \left| \int_C \left[f(z) - \sum_{k=0}^{n} a_k z^k \right] dz \right| = 0 \tag{2}$$

Since C lies inside the circle of convergence, the series converges uniformly on C to $f(z)$. For any ϵ, there is an $N(\epsilon)$ so that, *for all z on C*,

$$n \geq N(\epsilon) \Rightarrow \left| f(z) - \sum_{k=0}^{n} a_k z^k \right| < \epsilon$$

By the triangle inequality for integrals and the above inequalities, *for* $n \geq N$,

$$\left| \int_C \left[f(z) - \sum_{k=0}^{n} a_k z^k \right] dz \right| \leq \epsilon \cdot (\text{length of } C)$$

Since ϵ is arbitrary, the limit in (2) is zero. ■